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Proper incorporation of the self-adjoint extension method to the Green function formalism: one-dimensional δ'-function potential case

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Abstract. The one-dimensional δ'-function potential is discussed in the framework of the Green function formalism without invoking perturbation expansion. It is shown that the energy-dependent Green function for this case is crucially dependent on the boundary conditions which are provided by the self-adjoint extension method. The most general Green function which contains four real self-adjoint extension parameters is constructed. The relation between the bare coupling constant and the self-adjoint extension parameter is also derived.

Since the Kronig–Penny model [1] has been successful for the description of the energy band in solid-state physics, the point interaction problem has been applied in the various branches of physics for a long time. Recently the two-dimensional δ-function potential has been of interest in the context of the Aharonov–Bohm (AB) effect of spin-$\frac{1}{2}$ particles [2, 3] in which the delta function occurs as the mathematical description of the Zeeman interaction of the spin with a magnetic flux tube. In [4] two different approaches, renormalization and self-adjoint extension [5, 6], are presented for this subject. More recently the same problem has been re-examined in the framework of the Green function formalism [7, 8]. In [8] the present author showed how to incorporate the self-adjoint extension method within the Green function formalism without invoking the perturbation expansion.

Unlike the two- and three-dimensional cases, one-dimensional point interaction provides a four-parameter family solution, characterized by the boundary conditions at $x = 0$:

$$\begin{aligned}\varphi(\epsilon) &= \omega a\varphi(-\epsilon) + \omega b\varphi'(-\epsilon)\\ \varphi'(\epsilon) &= \omega c\varphi(-\epsilon) + \omega d\varphi'(-\epsilon)\end{aligned} \tag{1}$$

where ϵ is an infinitesimal positive parameter and $\omega \in \mathbb{C}$; $a, b, c, d \in \mathbb{R}$, satisfying $|\omega| = 1$ and $ad - bc = 1$ [6, 9]. Recently the path integral for the one-dimensional δ'-function potential has been calculated by incorporating Neumann boundary conditions within the usual perturbation theory of one-dimensional Dirac particle in order for the coupling constant to be infinitely repulsive [10].

In this paper we will discuss the one-dimensional δ'-function potential in the framework of the Green function formalism without using a perturbation expansion. It will be shown that the energy-dependent Green function is crucially dependent on the boundary conditions which are provided by the self-adjoint extension method in the present formalism. Choosing the boundary condition

$$\begin{aligned}\varphi'(\epsilon) &= \varphi'(-\epsilon) = \varphi'(0)\\ \varphi(\epsilon) - \varphi(-\epsilon) &= \beta\varphi'(0)\end{aligned} \tag{2}$$

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which is easily obtained from equation (1) by requiring $c = 0$, $\omega = a = d = 1$, and $b = \beta$, one can derive a similar result to that in [10].

However, the advantage of this formalism presented here is that one is free to choose boundary conditions. This means that one can get more general Green functions by choosing more general boundary conditions. If one chooses the most general boundary conditions (1) of the one-dimensional point interaction, the most general Green function, in which four real self-adjoint extension parameters are contained, can be derived. It is worthwhile noting that this formalism does not use the complicated perturbation expansion. Therefore, the calculation is very simple and clear.

Now let us start with one-dimensional system whose Hamiltonian is

$$H = H_0 + v\delta'(x) \tag{3}$$

where v is the bare coupling constant. Although H_0 can involve an arbitrary potential, in this paper we will only consider the free particle case for simplicity:

$$H_0 = \frac{p^2}{2}. \tag{4}$$

It is well known that the time-dependent Brownian motion propagator for the Hamiltonian (3) obeys integral equation [11–13]

$$G[x, y; t] = G_0[x, y; t] - v\int_0^t \mathrm{d}s \int \mathrm{d}z\, G_0[x, z; t-s]\delta'(z)G[z, y; s]. \tag{5}$$

After integrating with respect to z in equation (5), one can show easily

$$\begin{aligned}\hat{G}[x, y; E] = \hat{G}_0[x, y; E] &+ v\left(\frac{\partial \hat{G}_0[x, z; E]}{\partial z}\right)_{z=0}\hat{G}[0, y; E] \\ &+ v\hat{G}_0[x, 0; E]\left(\frac{\partial \hat{G}[z, y; E]}{\partial z}\right)_{z=0}.\end{aligned} \tag{6}$$

Equation (6) is purely formal. This is easily deduced from the fact that $\hat{G}[0, y; E]$ is not well defined because of the factor $|x|$ which is contained in $(\partial\hat{G}_0[x, z; E]/\partial z)_{z=0}$. Therefore, at this stage one has to conjecture the modifications of equation (6). Our conjecture for the modification of equation (6) is simply to extract the problematic zero point at $\hat{G}[x, y; E]$ as follows:

$$\begin{aligned}\hat{G}[x, y; E] = \hat{G}_0[x, y; E] &+ v\left(\frac{\partial \hat{G}_0[x, z; E]}{\partial z}\right)_{z=0}\hat{G}[\epsilon, y; E] \\ &+ v\hat{G}_0[x, 0; E]\left(\frac{\partial \hat{G}[z, y; E]}{\partial z}\right)_{z=\epsilon} \qquad \text{for } x > 0 \\ \hat{G}[x, y; E] = \hat{G}_0[x, y; E] &+ v\left(\frac{\partial \hat{G}_0[x, z; E]}{\partial z}\right)_{z=0}\hat{G}[-\epsilon, y; E] \\ &+ v\hat{G}_0[x, 0; E]\left(\frac{\partial \hat{G}[z, y; E]}{\partial z}\right)_{z=-\epsilon} \qquad \text{for } x < 0\end{aligned} \tag{7}$$

which might be a natural modification of equation (6).

In equation (7) the infinitesimal positive parameter ϵ is introduced. By inserting $x = \pm\epsilon$ in the first and second equations of equation (7) respectively, one can derive

$$\begin{aligned}\left(\frac{\partial \hat{G}[z, y; E]}{\partial z}\right)_{z=\epsilon} &= \frac{\sqrt{2E}}{v}[(1-v)\hat{G}[\epsilon, y; E] - \hat{G}_0[0, y; E]] \\ \left(\frac{\partial \hat{G}[z, y; E]}{\partial z}\right)_{z=-\epsilon} &= \frac{\sqrt{2E}}{v}[(1+v)\hat{G}[-\epsilon, y; E] - \hat{G}_0[0, y; E]].\end{aligned} \tag{8}$$

By inserting equation (8) into equation (7) $\hat{G}[x, y; E]$ becomes

$$\hat{G}[x, y; E] = \hat{G}_0[x, y; E] - \frac{1}{\sqrt{2E}} \mathrm{e}^{-\sqrt{2E}(|x|+|y|)} + \mathrm{e}^{-\sqrt{2E}|x|} \hat{G}[\epsilon, y; E] \qquad \text{for } x > 0$$

$$\hat{G}[x, y; E] = \hat{G}_0[x, y; E] - \frac{1}{\sqrt{2E}} \mathrm{e}^{-\sqrt{2E}(|x|+|y|)} + \mathrm{e}^{-\sqrt{2E}|x|} \hat{G}[-\epsilon, y; E] \qquad \text{for } x < 0. \quad (9)$$

Note that in equation (9) the v-dependence of $\hat{G}[x, y; E]$ is hidden in $\hat{G}[\pm\epsilon, y; E]$.

Now it is time to incorporate the self-adjoint extension method into the Green function formalism. First let us consider the simple boundary conditions given in equation (2). By applying these two boundary conditions to $\hat{G}[x, y; E]$, one can show that the boundary conditions generate two independent equations:

$$\hat{G}[\epsilon, y; E] + \hat{G}[-\epsilon, y; E] = \frac{2}{\sqrt{2E}} \mathrm{e}^{-\sqrt{2E}|y|}$$

$$\hat{G}[\epsilon, y; E] - \hat{G}[-\epsilon, y; E] = \beta[(\epsilon(y) + 1)\mathrm{e}^{-\sqrt{2E}|y|} - \sqrt{2E}\hat{G}[\epsilon, y; E]] \quad (10)$$

where

$$\epsilon(y) = \frac{y}{|y|}.$$

Therefore, by solving equation (10) the solutions

$$\hat{G}[\epsilon, y; E] = \frac{1}{\sqrt{2E}} \mathrm{e}^{-\sqrt{2E}|y|} \left(1 + \frac{\sqrt{2E}}{\sqrt{2E} + \frac{2}{\beta}} \epsilon(y)\right)$$

$$\hat{G}[-\epsilon, y; E] = \frac{1}{\sqrt{2E}} \mathrm{e}^{-\sqrt{2E}|y|} \left(1 - \frac{\sqrt{2E}}{\sqrt{2E} + \frac{2}{\beta}} \epsilon(y)\right) \quad (11)$$

are easily obtained. By combining equations (9) and (11) we get the final result

$$\hat{G}[x, y; E] = \hat{G}_0[x, y; E] + \frac{\epsilon(x)\epsilon(y)}{\sqrt{2E} + \frac{2}{\beta}} \mathrm{e}^{-\sqrt{2E}(|x|+|y|)}. \quad (12)$$

The relation between the bare coupling constant v and the self-adjoint extension parameter β is also obtained by inserting equation (11) into equation (8) and using the continuity of $\partial\hat{G}[z, y; E]/\partial z$ at $z = 0$. Unlike the two- and three-dimensional cases the relation is dependent on the space:

$$\frac{1}{v} = \left(1 + \sqrt{\frac{2}{E}} \frac{1}{\beta}\right) \qquad \text{for } y > 0$$

$$\frac{1}{v} = -\left(1 + \sqrt{\frac{2}{E}} \frac{1}{\beta}\right) \qquad \text{for } y < 0. \quad (13)$$

After taking the inverse Laplace transform of equation (12) and using analytic continuation in time, one can obtain the Feynman propagator (or kernel) $K[x, y; t]$:

$$K[x, y; t] = \frac{1}{\sqrt{2\pi \mathrm{i} t}} \exp\left(\frac{\mathrm{i}}{2t}|x - y|^2\right) + \frac{1}{\sqrt{2\pi \mathrm{i} t}} \exp\left(\frac{\mathrm{i}}{2t}(|x| + |y|)^2\right) \epsilon(x)\epsilon(y)$$
$$-\frac{1}{\beta} \exp\left(\frac{2}{\beta}(|x| + |y|) + \frac{2\mathrm{i}t}{\beta^2}\right) \mathrm{erfc}\left[\frac{1}{\sqrt{2\mathrm{i}t}}\left[|x| + |y| + \frac{2\mathrm{i}t}{\beta}\right]\right] \epsilon(x)\epsilon(y) \quad (14)$$

where erfc(z) is usual error function. Equation (14) coincides with equation (14) in [10] if one changes the β in [10] into $-\beta/2$. Therefore we have derived a similar result to that in [10] without invoking the perturbation expansion. Furthermore, in this formalism one can derive a more general Green function (or propagator) by using more general boundary conditions. Therefore, let us use the most general boundary condition (1) of the one-dimensional point interaction. In the same way as before these two boundary conditions provide two independent equations:

$$\left(\frac{d}{b}+\sqrt{2E}\right)\hat{G}[\epsilon,y;E]-\frac{\omega}{b}\hat{G}[-\epsilon,y;E]=(\epsilon(y)+1)\mathrm{e}^{-\sqrt{2E}|y|}$$
$$\frac{\omega^*}{b}\hat{G}[\epsilon,y;E]-\left(\sqrt{2E}+\frac{a}{b}\right)\hat{G}[-\epsilon,y;E]=(\epsilon(y)-1)\mathrm{e}^{-\sqrt{2E}|y|} \tag{15}$$

where ω^* is the complex conjugate of ω. By inserting the solutions of equation (15)

$$\hat{G}[\epsilon,y;E]=-\frac{\mathrm{e}^{-\sqrt{2E}|y|}}{\frac{c}{b}+\sqrt{2E}\frac{a+d}{b}+2E}\left[\epsilon(y)\left(\frac{\omega}{b}-\sqrt{2E}-\frac{a}{b}\right)-\left(\frac{\omega}{b}+\sqrt{2E}+\frac{a}{b}\right)\right]$$
$$\hat{G}[-\epsilon,y;E]=-\frac{\mathrm{e}^{-\sqrt{2E}|y|}}{\frac{c}{b}+\sqrt{2E}\frac{a+d}{b}+2E}\left[\epsilon(y)\left(\frac{d}{b}+\sqrt{2E}-\frac{\omega^*}{b}\right)-\left(\frac{d}{b}+\sqrt{2E}+\frac{\omega^*}{b}\right)\right] \tag{16}$$

into equation (9) it is straightforward to derive the energy-dependent Green function corresponding to the most general boundary conditions:

$$\begin{aligned}\hat{G}[x,y;E]=\hat{G}_0[x,y;E]+\frac{\sqrt{2E}b}{D(E)}\mathrm{e}^{-\sqrt{2E}(|x|+|y|)}\epsilon(x)\epsilon(y)-\frac{\mathrm{e}^{-\sqrt{2E}(|x|+|y|)}}{D(E)}\\ \times\left[\frac{c}{\sqrt{2E}}+\frac{1}{2}(a+d-\omega-\omega^*)+\frac{1}{2}(d-a+\omega^*-\omega)\epsilon(x)\right.\\ \left.+\frac{1}{2}(d-a+\omega-\omega^*)\epsilon(y)-\frac{1}{2}(a+d-\omega-\omega^*)\epsilon(x)\epsilon(y)\right]\end{aligned} \tag{17}$$

where

$$D(E)=c+(a+d)\sqrt{2E}+2Eb. \tag{18}$$

Note that equation (17) coincides with equation (12) at $c=0$, $\omega=a=d=1$, and $b=\beta$. The energy-dependent Green function for the one-dimensional point interaction is calculated in [14]. The result (17) is exactly the same as that in [14] although the authors of [14] claimed that their result is a consequence of the appropriate mixture of one-dimensional δ- and δ'-potentials. In this paper the same result can be derived by using only the one-dimensional δ'-function potential. Of course by following the procedure presented in [14] one can also obtain the time-dependent Brownian motion propagator and Feynman kernel straightforwardly. One can also derive the relation between the bare coupling constant and the self-adjoint parameters as before.

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